Maintaining a Bounded Degree Expander in Dynamic Peer-to-Peer Networks

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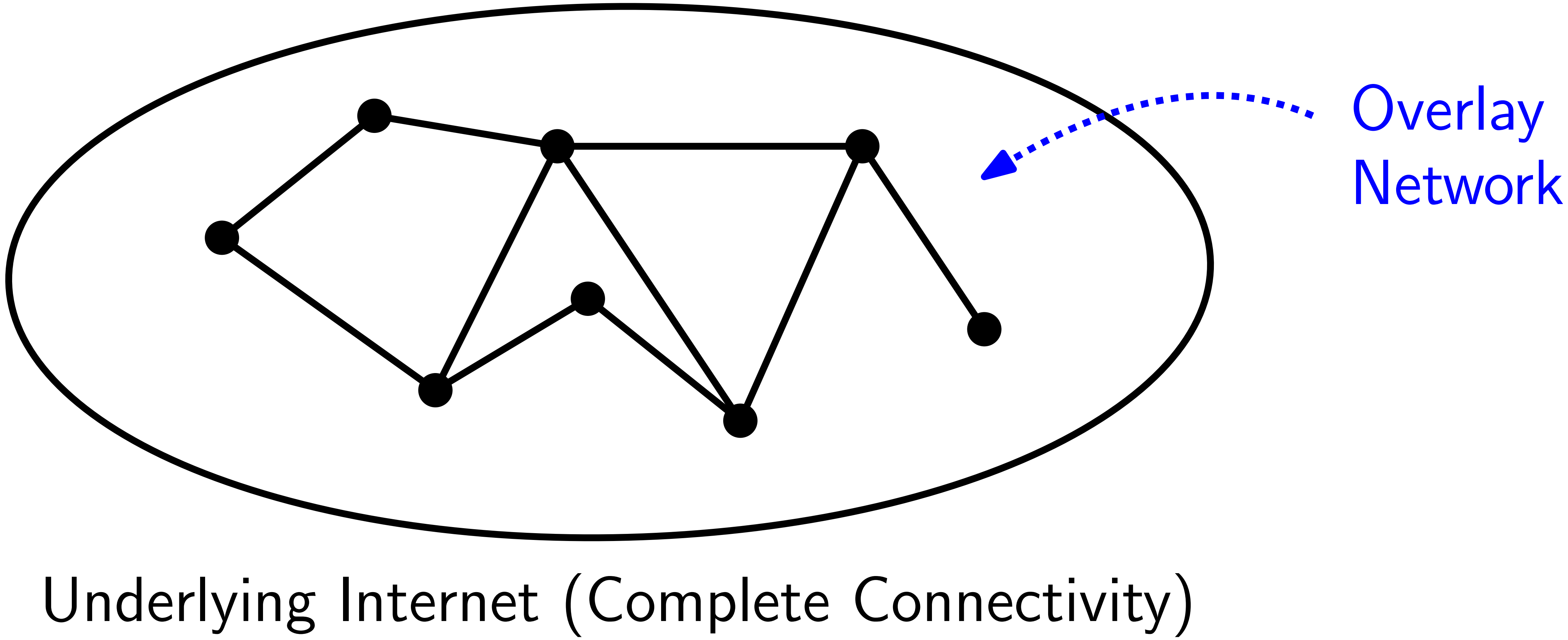
Peer to Peer Networks

Prevailing Definition: A network of peers, ideally fully decentralized

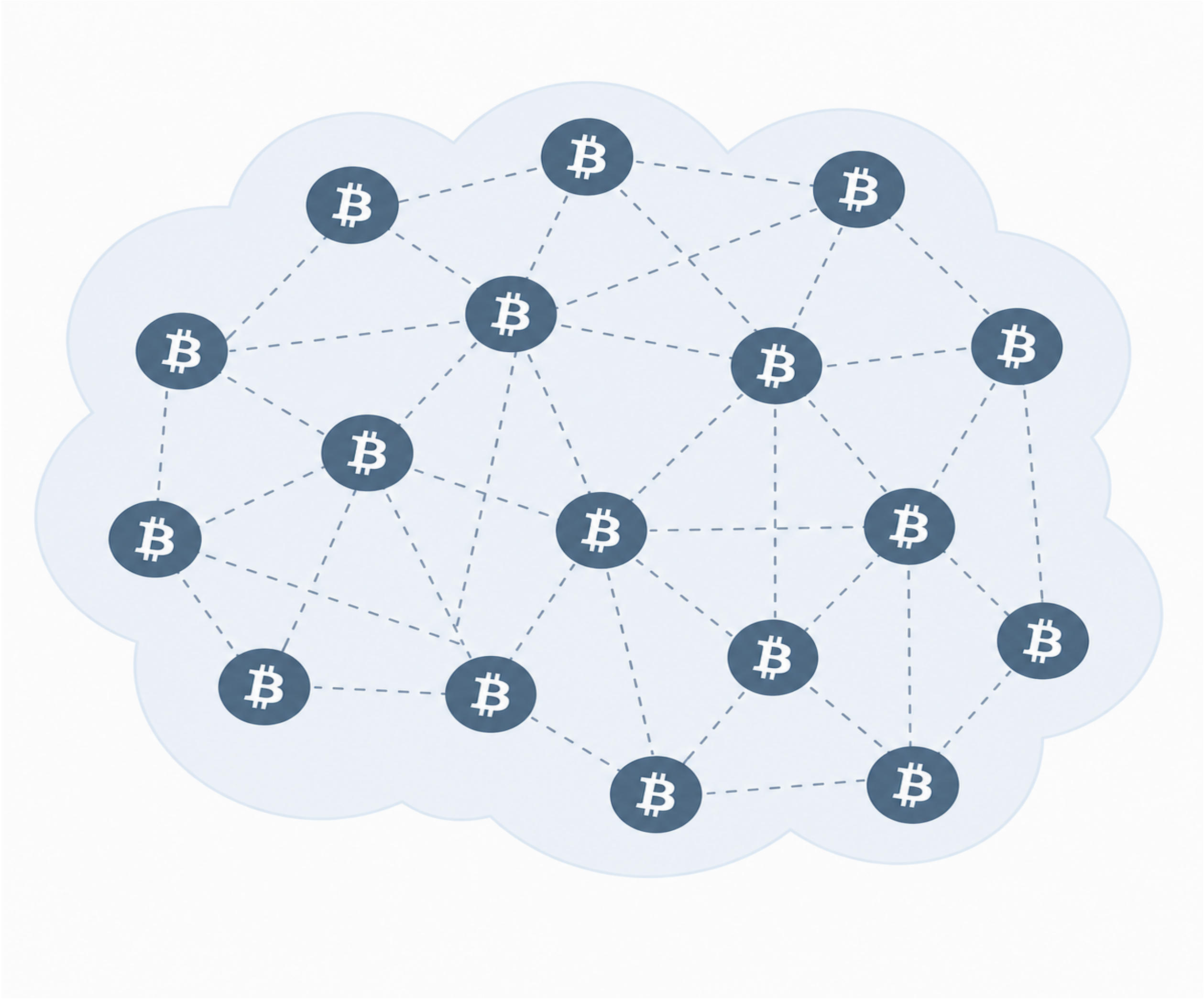
Key Challenges:

- Highly dynamic
- High **churn**



Motivation: the Bitcoin network

We can approximately retrieve the set of peers



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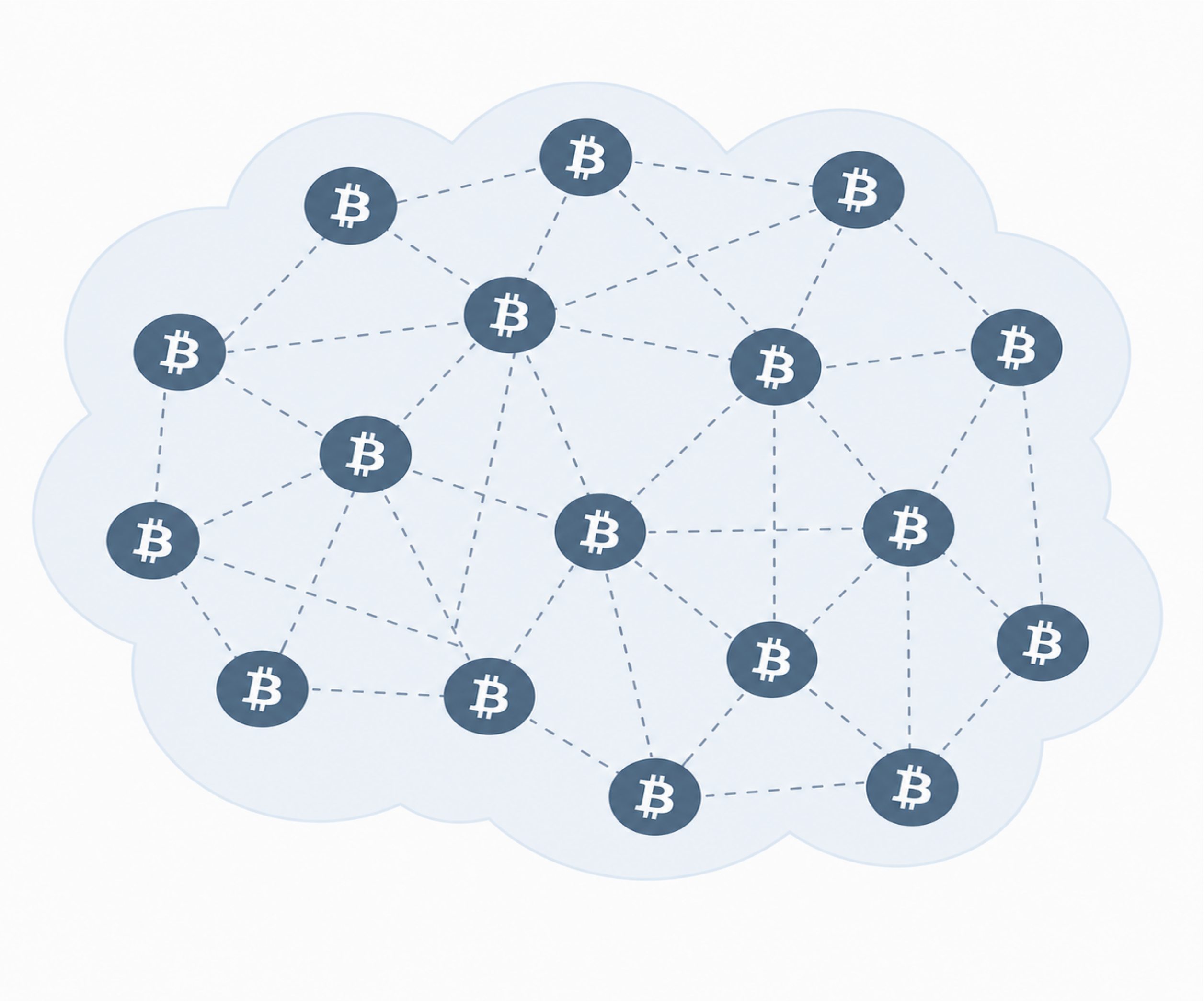

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But: we can't find the links between them

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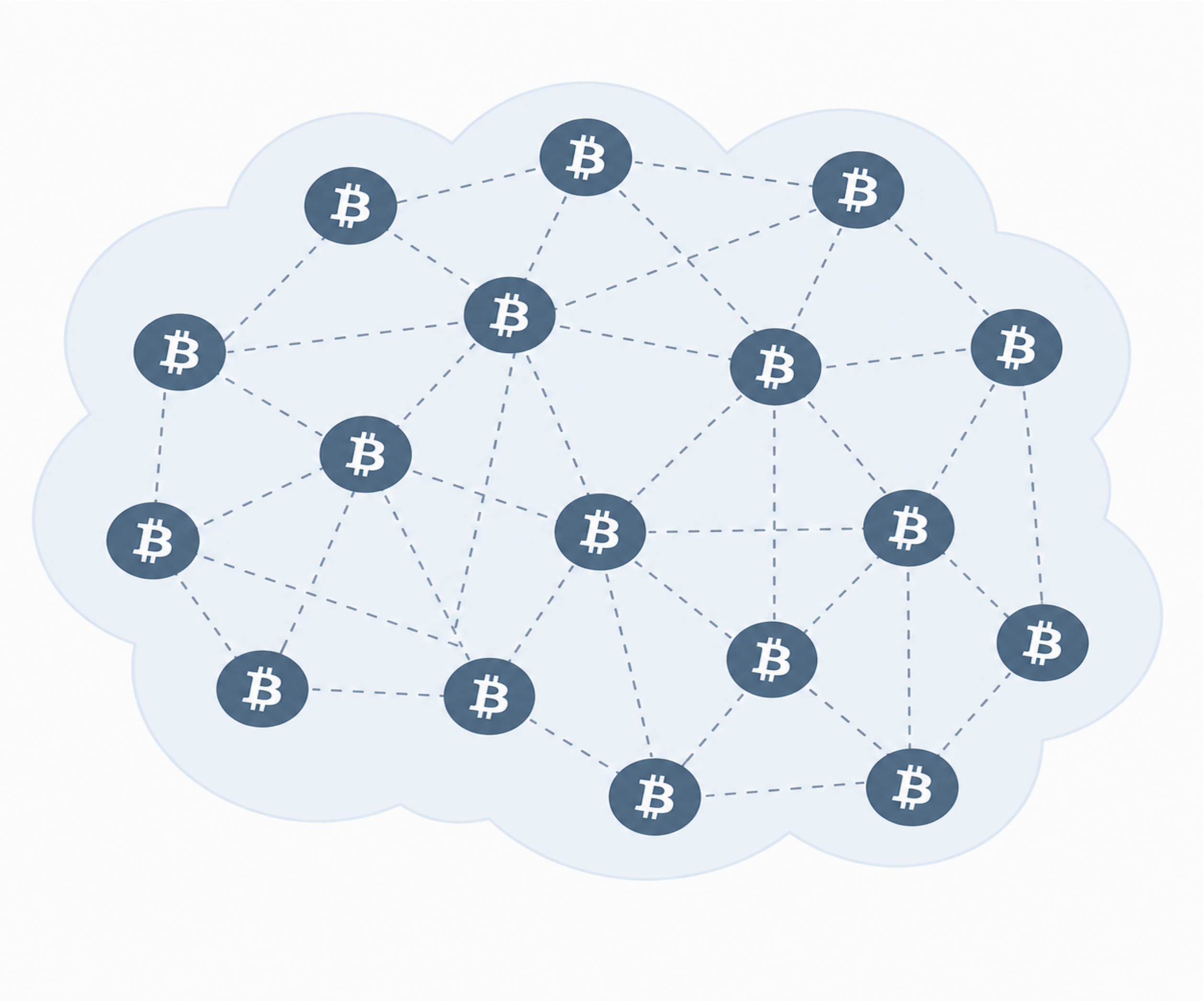

But: we can't find the links between them

What do we know?

Every node must have:

- Min. 8 connections
- Max. 125 connections

RAES(d, Δ) [Becchetti, Clementi, Natale, Pasquale, Trevisan SODA '20]

Let $G_0 = (V, \{\emptyset\})$.
For $t = 0, 1, 2, \ldots$

- **Phase 1:** Each node u with degree $d(u) < d$ picks $d - d(u)$ new neighbors uniformly at random.

- **Phase 2:** Each node u with degree $d(u) > \Delta$ selects $d(u) - \Delta$ neighbors uniformly at random and drops the connection.

Converges to a <u>**static**</u> bounded degree expander graph in $\mathcal{O}(\log n)$ rounds whp.

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Question: What happens if nodes can join and leave the network?

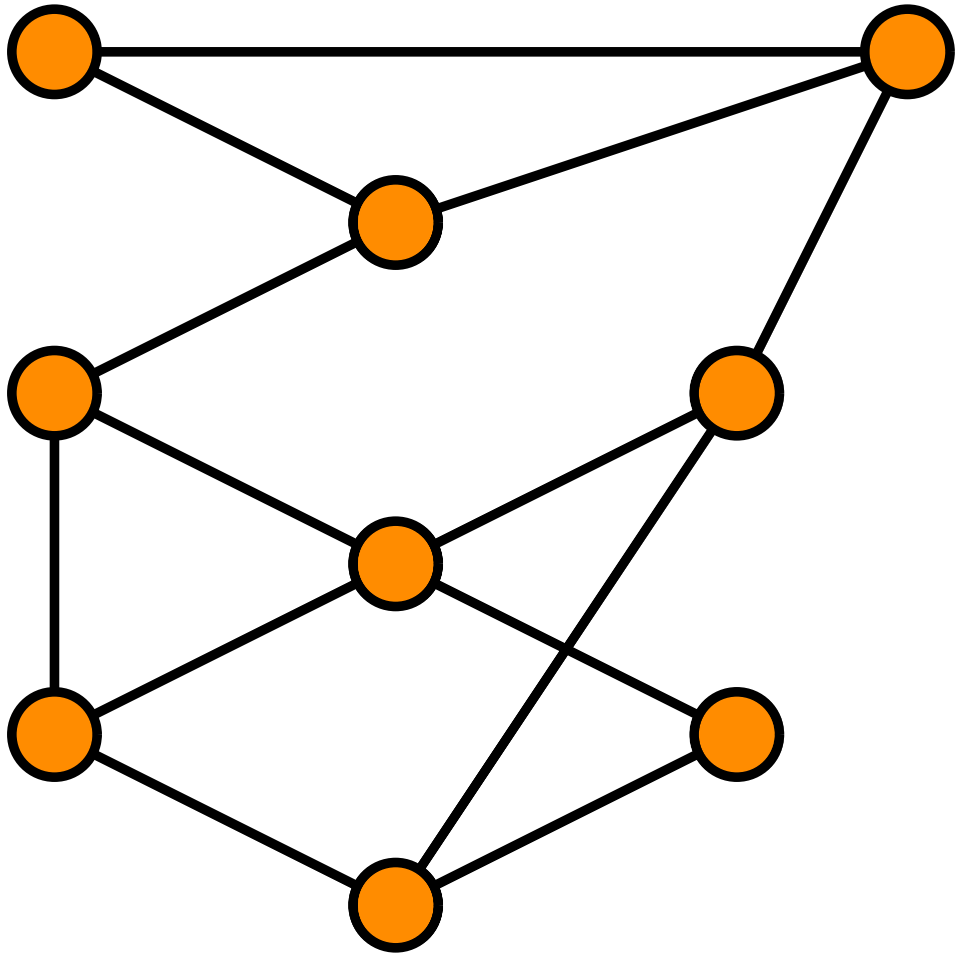
Related Work

- Empirical evidence that RAES can tolerate churn [Cruciani & Pasquale ICDCN'23]

- Theoretical Analysis of the RAES protocol in the Streaming node-churn model (deterministic churn) [Angileri, Clementi, Natale, Salvi, Ziccardi STACS'26]

Our setting: oblivious adversarial churn

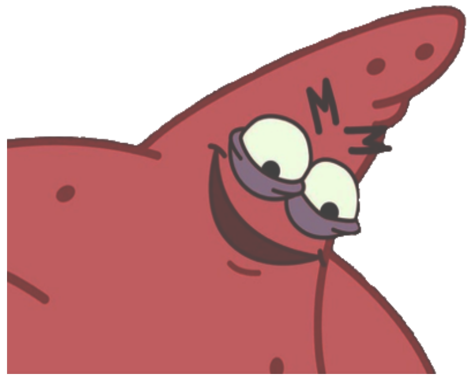
At round t

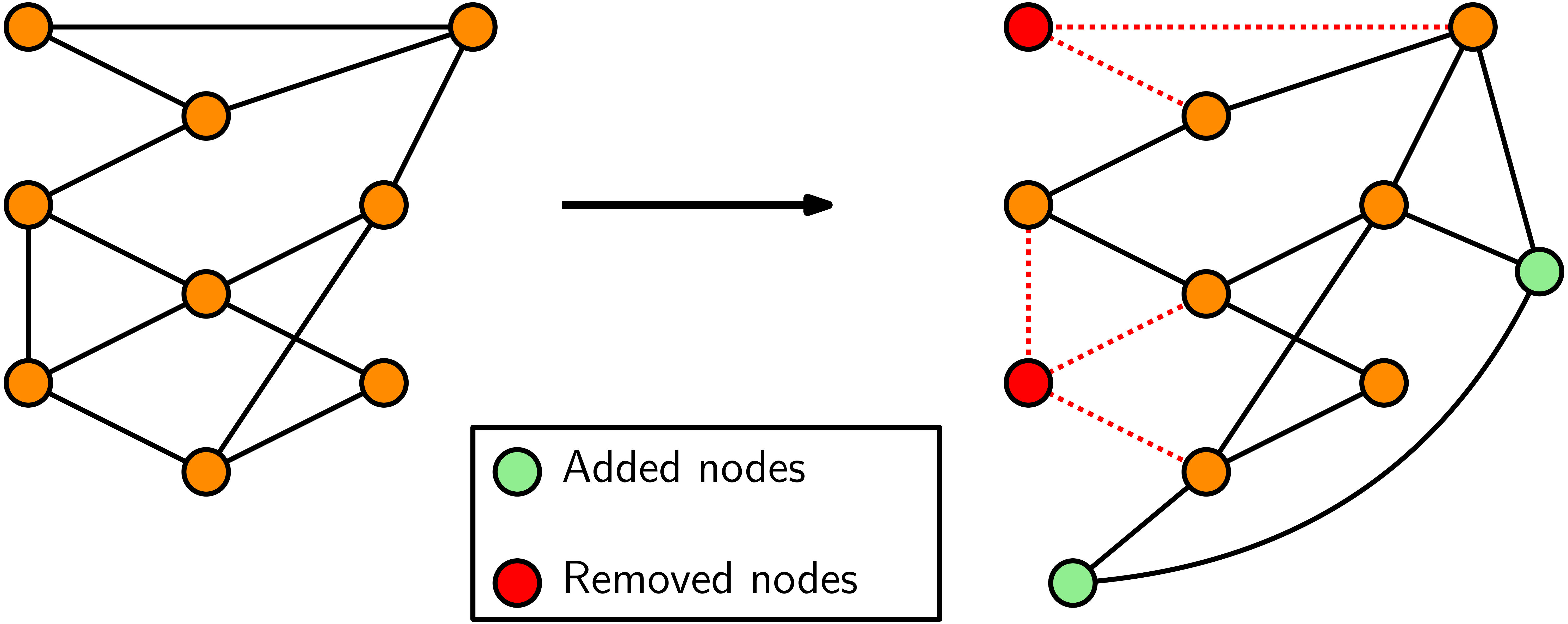


Our setting: oblivious adversarial churn

Oblivious adversary

At round t





Model: Dynamic Network with Churn (DNC)

Synchronous: All nodes follows the same clock. In each round $t = 1, 2, 3, \ldots$

- each node that joined the network has access to the uniform distribution over V_t

Adversarial Dynamism:

An **oblivious** adversary (knows the algorithm but not the coin toss outcomes) designs the churn

$$\mathcal{G} = \left(G^0, G^1, \ldots, G^t, \ldots\right)$$

- Must "attach" each joining node at time t with at least one pre-existing node.
- Keep the degree $\texttt{deg}(G^t) \leq \delta$

Bootstrap and Maintenance Phases

$\mathcal{O}(\log n)$ rounds Bootstrap Phase

Algorithm initialization

Adversary wakes up

Churn rate of up to $\mathcal{O}(n/\text{polylog}(n))$ per round

Maintenance Phase

We need to cope with the churn

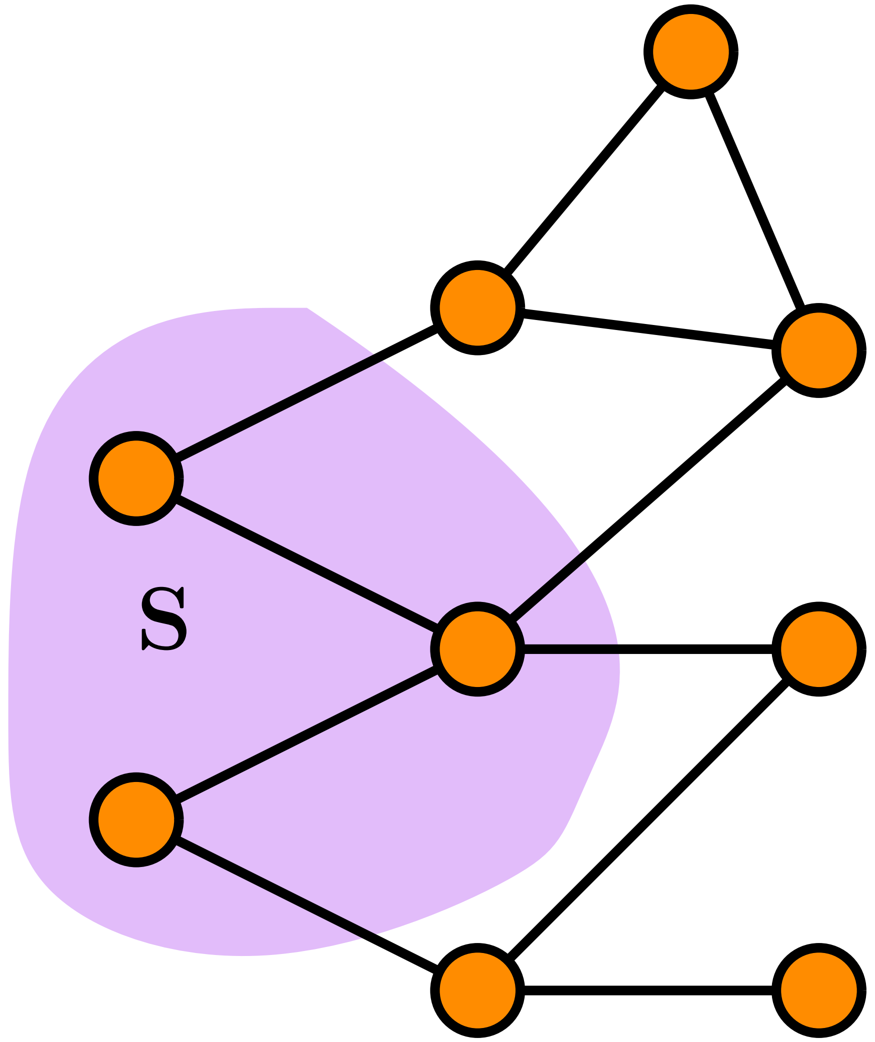
Preliminaries

Edge Expansion of a set $\mathbf{S} \subseteq \mathbf{V}$

$$\phi(S) = \frac{\#\text{ Edges between } S \text{ and } \overline{S}}{|S|}$$

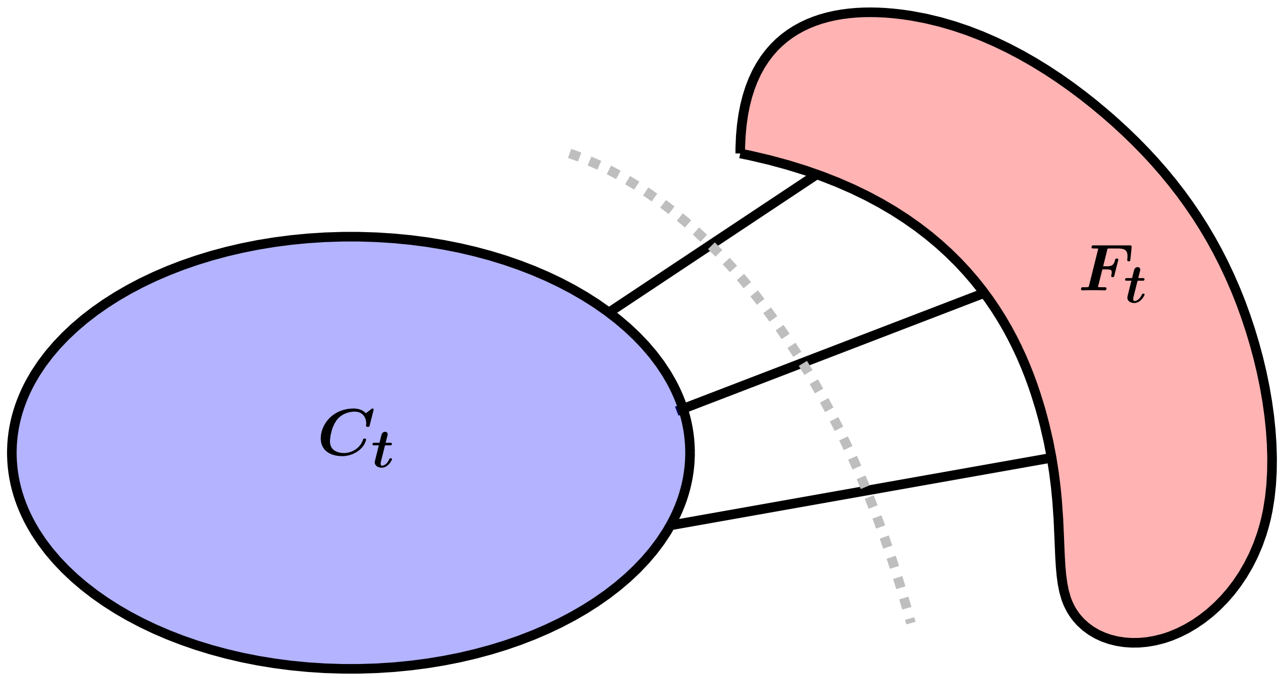
Edge Expansion of a graph $\mathbf{G}$

$$\phi(G) = \min_{\substack{S \subseteq V \\ 0 < |S| \leq n/2}} \phi(S)$$



Observation

Problem: There is an adversarial strategy that deteriorates the expansion whp.



Solution:

Phase 0 (refresh neighbors): With probability $1/\text{polylog}(n)$ each node u with degree $d(u) \in [d, \Delta]$ drops all its neighbors.

The D-RAES

Let $G_0 = (V, \{\emptyset\})$.
For $t = 0, 1, 2, \ldots$

- **Phase 0:** With probability $1/\text{polylog}(n)$ each node u with degree $d(u) \in [d, \Delta]$ drops all its neighbors.

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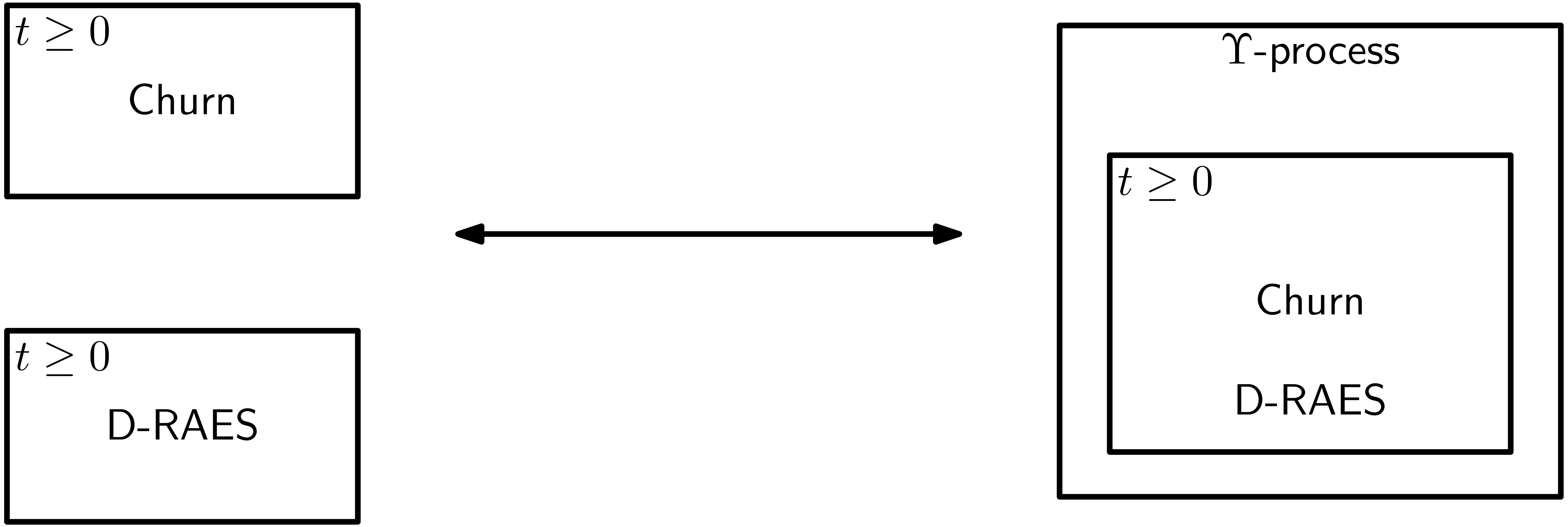
The Υ-Process



Theorem 1 (Main theorem, informal) *Under adversarial churn of* $\mathcal{O}(n/\log^k n)$ *nodes per round, D-RAES maintains, with high probability, a subgraph* C_t *on* $n - o(n)$ *nodes, with degree in* $[d, \Delta]$ *and constant edge expansion.*

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Let D_t be the set of vertices whose local structure may be corrupted during round t. This includes:

$$D_t = D_t^{\text{churn}} \cup D_t^{\text{refresh}} \cup D_t^{\text{prune}}.$$

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We show that each part is small.

- The adversary churns only $\mathcal{O}(n/\log^k n)$ vertices.
- By Chernoff bounds, only $\mathcal{O}(n/\log^k n)$ vertices refresh their neighborhoods in round t.
- The random reconnection and pruning phases create only $\mathcal{O}(n/\log^k n)$ additional vertices whose degree may fall below d, with high probability.

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- remove D_t from G_{t-1}.

- Deleting $o(n)$ vertices from an α-expander leaves a subgraph C_t of size $n - o(n)$ whose edge expansion is still constant [Bagchi, Bhargava, Chaudhary, Eppstein, Scheideler, TCS '06].

Future work

- What if the nodes have value of d and/or Δ drawn from a power-law distribution?
- What about stochastic churn?

Thank You