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Expander Graphs	
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Expanders are sparse graphs that behave, in many important ways, like complete graphs. A complete graph is extremely well connected, but it has quadratically many edges. An expander keeps only a linear number of edges while still having no sparse cuts: every set of vertices has many edges leaving it. This makes expanders a central object in theoretical computer science, with applications ranging from network design and error reduction to pseudorandomness, derandomization, and distributed computation.

1 Edge ad Vertex Expansion of a graph

Definition 1 (Edge expansion of a set)**.** *Let $G = (V, E)$ be a d-regular graph, and $S \subseteq V$ a subset of vertices. The edge expansion of S is*

$$\phi(S) = \frac{|E(S, \overline{S})|}{d|S|}$$

where $|E(S, \overline{S})|$ is the number of edges in E that have one endpoint in S and one endpoint in $\overline{S} = V \setminus S$.

observe that $d|S|$ is a trivial upper bound to the number of edges that can leave S, and so $\phi(S)$ measures how much smaller the actual number of edges is than this upper bound. Another way to see it is to think of $\phi(S)$ as the probability that, if we pick a random node v in S and then a random neighbor of w of v, the node w happens to be outside S.

Observe that $1 - \phi(S)$ is the average fraction of neighbors that nodes in S have within S. For example, if our graph G is a social network, and S a subset of users of expansion κ, this means that, on average, the users in S have $(1 - \kappa)\%$ of their friends within S.

If $(S, V \setminus S)$ is a cut of the graph, and $|S| \leq |V \setminus S|$, then $\phi(S)$ is, within factor 2, the ratio between

$$\frac{|E(S, V \setminus S)|}{|E|} = \frac{2\,|E(S, V \setminus S)|}{dn}$$

of edges that we have to remove to disconnect S from $V \setminus S$, and the fraction

$$|S| \cdot \frac{|V \setminus S|}{\binom{n}{2}}$$

of pairs of vertices that become unreachable if we do so. We define the edge expansion of a cut as

$$\phi(S, V \setminus S) = \max\{\phi(S), \phi(V \setminus S)\} = \max\{\phi(S), \phi(\overline{S})\}.$$

The edge expansion of a graph is the minimum of the edge expansion of all cuts in the graph.

Definition 2 (Edge Expansion)**.** *Let $G = (V, E)$ be a d-regular graph, its edge epxansion is*

$$\phi(G) = \min_{\substack{S \subseteq V \\ 0<|S|<|V|}} \phi(S, \overline{S}) = \min_{\substack{S \subseteq V \\ 0<|S|\leq \frac{|V|}{2}}} \phi(S)$$

A family of constant-degree expanders is a collection of arbitrarily large graphs, all of degree $\mathcal{O}(1)$ and edge expansion $\Omega(1)$. Expanders are useful in several applications, and a common theme in such applications is that even though they are sparse, they have some of the "connectivity" properties of a complete graph.

For example, if one removes a $o(1)$ fraction of edges from an expander, one is left with a connected component that contains a $1 - o(1)$ fraction of vertices.

Lemma 3. *Let $G = (V, E)$ be a regular graph of expansion ϕ. Then, after an $\varepsilon < \phi$ fraction of the edges are adversarially removed, the graph has a connected component that spans at least*

$$1 - \frac{\varepsilon}{2\phi}$$

fraction of the vertices.

Proof. Let

$$C_1, C_2, \ldots, C_m$$

be the connected components of the graph $H = (V, E \setminus E')$, ordered so that

$$|C_1| \geq |C_2| \geq \cdots \geq |C_m|.$$

We want to show that C_1 is large.

Since G is d-regular, we have

$$|E| = \frac{d|V|}{2}.$$

Therefore the number of removed edges satisfies

$$|E'| \leq \varepsilon|E| = \varepsilon\frac{d|V|}{2}.$$

We first observe that every edge of G going between two distinct components C_i and C_j must belong to E'. Indeed, if an edge between C_i and C_j had not been removed, then C_i and C_j would be connected in $H = (V, E \setminus E')$, contradicting the fact that they are distinct connected components.

Hence,

$$|E'| \geq \frac{1}{2}\sum_{i=1}^{m} |E(C_i, V \setminus C_i)|.$$

The factor $1/2$ appears because each edge between two different components is counted twice in the sum: once from the side of one component, and once from the side of the other.

We now prove that C_1 must contain more than half of the vertices. Suppose, for contradiction, that

$$|C_1| \leq \frac{|V|}{2}.$$

Since C_1 is the largest connected component, this implies that

$$|C_i| \leq \frac{|V|}{2} \qquad \text{for every } i = 1, \dots, m.$$

By the definition of edge expansion, for every such component C_i, we have

$$|E(C_i, V \setminus C_i)| \geq d\phi |C_i|.$$

Therefore,

$$|E'| \geq \frac{1}{2} \sum_{i=1}^{m} |E(C_i, V \setminus C_i)| \geq \frac{1}{2} \sum_{i=1}^{m} d\phi |C_i|.$$

Since the sets $C_1, \dots, C_m$ form a partition of V, we have

$$\sum_{i=1}^{m} |C_i| = |V|.$$

Thus,

$$|E'| \geq \frac{1}{2} d\phi |V|.$$

On the other hand, by assumption,

$$|E'| \leq \varepsilon \frac{d|V|}{2}.$$

Combining the two inequalities gives

$$\varepsilon \frac{d|V|}{2} \geq \phi \frac{d|V|}{2},$$

and hence

$$\varepsilon \geq \phi,$$

which contradicts the assumption $\varepsilon < \phi$.

Therefore,

$$|C_1| > \frac{|V|}{2}.$$

Now define

$$S = V \setminus C_1 = C_2 \cup \cdots \cup C_m.$$

Since $|C_1| > |V|/2$, we have

$$|S| < \frac{|V|}{2}.$$

Again by the definition of edge expansion,

$$|E(S, V \setminus S)| \geq d\phi |S|.$$

But $V \setminus S = C_1$, so

$$|E(S, V \setminus S)| = |E(S, C_1)|.$$

Every edge between S and C_1 must have been removed; otherwise S and C_1 would not be separated into different connected components of $H = (V, E \setminus E')$. Therefore,

$$|E'| \geq |E(S, C_1)|.$$

Combining the last inequalities, we get

$$|E'| \geq |E(S, C_1)| = |E(S, V \setminus S)| \geq d\phi|S|.$$

Using again the upper bound on the number of removed edges,

$$d\phi|S| \leq |E'| \leq \varepsilon \frac{d|V|}{2}.$$

Dividing by $d\phi$, we obtain

$$|S| \leq \frac{\varepsilon}{2\phi}|V|.$$

Since $S = V \setminus C_1$, this implies

$$|C_1| = |V| - |S| \geq |V| - \frac{\varepsilon}{2\phi}|V| = \left(1 - \frac{\varepsilon}{2\phi}\right)|V|.$$

Therefore the graph $H = (V, E \setminus E')$ contains a connected component spanning at least

$$1 - \frac{\varepsilon}{2\phi}$$

fraction of the vertices. □

Proposition 4. *Let $G = (V, E)$ be a d-regular graph on n vertices with normalized edge expansion $\phi > 0$. Then*

$$\operatorname{diam}(G) \leq 2\left\lceil \log_{1+\phi} n \right\rceil.$$

In particular, if $\phi = \Omega(1)$, then

$$\operatorname{diam}(G) = \mathcal{O}(\log n).$$

Proof. For a set $S \subseteq V$, define its closed neighborhood as

$$N[S] = S \cup \{v \in V : \text{there exists } u \in S \text{ such that } \{u, v\} \in E\}.$$

We first show that every set of size at most $n/2$ expands by a multiplicative factor. Let $S \subseteq V$ satisfy

$$0 < |S| \leq \frac{n}{2}.$$

By the definition of normalized edge expansion,

$$|E(S, V \setminus S)| \geq d\phi|S|.$$

Every vertex outside S can be incident to at most d edges coming from S. Therefore, the number of vertices outside S that have at least one neighbor in S is at least

$$\frac{|E(S, V \setminus S)|}{d} \geq \phi|S|.$$

Hence

$$|N[S]| = |S| + |N[S] \setminus S| \geq |S| + \phi|S| = (1 + \phi)|S|.$$

Now fix a vertex $v \in V$, and let

$$B_r(v) = \{u \in V : \mathrm{dist}(u, v) \leq r\}$$

be the ball of radius r around v. Since

$$B_{r+1}(v) = N[B_r(v)],$$

the previous argument implies that, as long as

$$|B_r(v)| \leq \frac{n}{2},$$

we have

$$|B_{r+1}(v)| \geq (1 + \phi)|B_r(v)|.$$

Since $B_0(v) = \{v\}$, it follows that the ball grows by a factor $1 + \phi$ at every step until it contains more than half of the graph.

Let

$$R = \lceil \log_{1+\phi} n \rceil .$$

Then, for every vertex v, we must have

$$|B_R(v)| > \frac{n}{2}.$$

Indeed, otherwise the multiplicative growth would imply

$$|B_R(v)| \geq (1 + \phi)^R \geq n,$$

which is impossible unless the ball already contains more than half of the vertices at some earlier step.

Now take any two vertices $u, v \in V$. Since both balls $B_R(u)$ and $B_R(v)$ have size larger than $n/2$, they must intersect. Therefore, there exists a vertex $w \in V$ such that

$$w \in B_R(u) \cap B_R(v).$$

Hence

$$\mathrm{dist}(u, v) \leq \mathrm{dist}(u, w) + \mathrm{dist}(w, v) \leq R + R = 2R.$$

Since this holds for every pair of vertices u, v, we conclude that

$$\mathrm{diam}(G) \leq 2 \lceil \log_{1+\phi} n \rceil .$$

Finally, if $\phi = \Omega(1)$, then

$$\log_{1+\phi} n = \mathcal{O}(\log n),$$

and therefore

$$\mathrm{diam}(G) = \mathcal{O}(\log n).$$

□

Thus expanders have small diameter: starting from any vertex, the ball around it grows by a constant factor at each step until it contains a constant fraction of the graph.

Example: The clique has constant expansion. Let $K_n = (V, E)$ be the complete graph on n vertices. This graph is $(n-1)$-regular.

Let $S \subseteq V$ be such that $0 < |S| \leq n/2$. Since K_n is complete, every vertex in S is connected to every vertex in $V \setminus S$. Therefore,

$$|E(S, \overline{S})| = |S|(n - |S|).$$

Since K_n is $(n-1)$-regular, we get

$$\phi(S) = \frac{|S|(n - |S|)}{(n-1)|S|} = \frac{n - |S|}{n-1}.$$

This quantity is minimized when $|S|$ is as large as possible, namely when

$$|S| = \left\lfloor \frac{n}{2} \right\rfloor.$$

Hence,

$$\phi(K_n) = \frac{n - \lfloor n/2 \rfloor}{n-1} = \frac{\lceil n/2 \rceil}{n-1}.$$

In particular,

$$\phi(K_n) = \Theta(1).$$

Thus the clique has constant edge expansion. This reflects the fact that the clique is extremely well connected: every set of at most half the vertices has a constant fraction of its incident edges leaving the set.

However, the clique is not a constant-degree graph, since its degree is $n-1$. Expander graphs can be viewed as sparse graphs that retain some of the strong connectivity properties of the clique while having only constant degree.

Example: The cycle is not an expander. Let $C_n = (V, E)$ be the cycle graph on n vertices. This graph is 2-regular. We show that the expansion of C_n is of order $1/n$.

Take S to be a set of consecutive vertices on the cycle with

$$|S| = \left\lfloor \frac{n}{2} \right\rfloor.$$

There are exactly two edges leaving S, namely the two edges at the boundary of the interval of consecutive vertices. Therefore,

$$|E(S, \overline{S})| = 2.$$

Since C_n is 2-regular, we get

$$\phi(S) = \frac{|E(S, \overline{S})|}{2|S|} = \frac{2}{2\lfloor n/2 \rfloor} = \frac{1}{\lfloor n/2 \rfloor}.$$

Thus,

$$\phi(C_n) \leq \frac{1}{\lfloor n/2 \rfloor}.$$

Conversely, let $S \subseteq V$ be any non-empty set with

$$|S| \le \frac{n}{2}.$$

Since $S \ne \emptyset$ and $S \ne V$, at least two edges of the cycle cross from S to $V \setminus S$. Hence

$$|E(S, \overline{S})| \ge 2.$$

Therefore,

$$\phi(S) = \frac{|E(S, \overline{S})|}{2|S|} \ge \frac{2}{2|S|} = \frac{1}{|S|} \ge \frac{1}{\lfloor n/2 \rfloor}.$$

Combining the upper and lower bounds, we obtain

$$\phi(C_n) = \frac{1}{\lfloor n/2 \rfloor}.$$

In particular,

$$\phi(C_n) = \Theta\left(\frac{1}{n}\right).$$

Thus the family of cycle graphs is not a family of expanders: as n grows, its edge expansion goes to zero.

2 Probabilistic construction of expander graphs

We now show, using the probabilistic method, that constant-degree expanders exist.

Assume for simplicity that n is even. We consider the following random multigraph $G = (V, E)$ on n vertices. Independently sample d uniformly random perfect matchings

$$M_1, \dots, M_d$$

on V, and let G be their union. Thus every vertex has degree exactly d, counting parallel edges.

The random model. Let n be even and let V be a set of n vertices. We generate a random d-regular multigraph as follows.

1. Independently sample d uniformly random perfect matchings

   $$M_1, \dots, M_d$$

   on the vertex set V.

2. Let

   $$E = M_1 \cup M_2 \cup \cdots \cup M_d,$$

   where parallel edges are kept with multiplicity.

3. Output the multigraph

   $$G = (V, E).$$

Since each perfect matching contributes exactly one incident edge to each vertex, the resulting multigraph is d-regular, counting parallel edges with multiplicity.

Sampling one random perfect matching. It will be useful to view a random perfect matching through the following equivalent exposure process. Fix an arbitrary ordering

$$v_1, \ldots, v_n$$

of the vertices.

1. Initialize
$$M = \emptyset, \qquad C = \emptyset,$$
where M is the matching constructed so far and C is the set of already matched vertices.

2. While $C \neq V$:
   (a) Let v be the first vertex in the ordering that does not belong to C.
   (b) Choose w uniformly at random from
   $$V \setminus (C \cup \{v\}).$$
   (c) Add the edge $\{v, w\}$ to the matching:
   $$M = M \cup \{\{v, w\}\}.$$
   (d) Mark both vertices as matched:
   $$C = C \cup \{v, w\}.$$

3. Return M.

This procedure samples a uniformly random perfect matching. Indeed, at the first step there are $n-1$ choices for the partner of the first vertex, then $n-3$ choices for the partner of the next unmatched vertex, and so on. Hence the procedure has

$$(n-1)(n-3)\cdots 3 \cdot 1$$

possible outputs, each occurring with the same probability. This is exactly the number of perfect matchings on n vertices.

Theorem 5. *There exists a constant d such that a random d-regular graph is an expander with high probability.*

More precisely, let G be obtained as the union of 48 independent uniformly random perfect matchings on an n-vertex set V, where n is even. Then, with probability at least $1 - \mathcal{O}(n^{-2})$,

$$\phi(G) \geq \frac{1}{288}.$$

Proof. Let $d = 48$. We sample G as the union of d independent uniformly random perfect matchings on V. Thus G is d-regular, counting parallel edges with multiplicity.

We prove that, with high probability, every set $S \subseteq V$ with

$$0 < |S| \le \frac{n}{2}$$

has at least

$$\frac{|S|}{6}$$

edges leaving it. Since G is d-regular, this implies

$$\phi(S) = \frac{|E(S, V \setminus S)|}{d|S|} \ge \frac{|S|/6}{d|S|} = \frac{1}{6d} = \frac{1}{288}.$$

Fix a set $S \subseteq V$ of size

$$|S| = k \le \frac{n}{2}.$$

Let $\Gamma(S)$ denote the set of vertices that have at least one neighbor in S.

Suppose that

$$|\Gamma(S) \setminus S| \le \frac{k}{6}.$$

Then all neighbors of S are contained in $S \cup T$ for some set

$$T \subseteq V \setminus S, \qquad |T| = \frac{k}{6}.$$

For simplicity, we ignore floors and ceilings. Replacing $k/6$ by $\lceil k/6 \rceil$ changes only the constants.

We now bound the probability that such a set T exists.

Fix sets S, T as above. First consider one uniformly random perfect matching M. Expose M sequentially as follows. Order the vertices so that the vertices of S come first. Repeatedly take the first unmatched vertex v, and match it to a uniformly random unmatched vertex w.

For all vertices of S to be matched inside $S \cup T$, during the first $k/2$ iterations the random partner w must always lie in $S \cup T$. Hence

$$\Pr(\text{all vertices of } S \text{ are matched inside } S \cup T) \le \prod_{i=1}^{k/2} \frac{\frac{7k}{6} - 2i + 1}{n - 2i + 1}.$$

Keeping only the last $k/6$ factors, namely

$$i = \frac{k}{3} + 1, \dots, \frac{k}{2},$$

we get

$$\prod_{i=1}^{k/2} \frac{\frac{7k}{6} - 2i + 1}{n - 2i + 1} \le \prod_{i=k/3+1}^{k/2} \frac{k/2}{n/2} = \left(\frac{k}{n}\right)^{k/6}.$$

Therefore, for one matching,

$$\Pr(\text{all vertices of } S \text{ are matched inside } S \cup T) \le \left(\frac{k}{n}\right)^{k/6}.$$

Since the d matchings are sampled independently, we obtain

$$\Pr(\Gamma(S) \subseteq S \cup T) \leq \left(\frac{k}{n}\right)^{dk/6}.$$

Now take a union bound over all choices of S and T. There are

$$\binom{n}{k}$$

choices for S, and at most

$$\binom{n}{k/6}$$

choices for T. Therefore,

$$\Pr\left(\exists S,\ |S| = k,\ |\Gamma(S) \setminus S| \leq \frac{k}{6}\right) \leq \left(\frac{k}{n}\right)^{dk/6} \binom{n}{k/6}\binom{n}{k}.$$

Since $k/6 \leq k \leq n/2$, we have

$$\binom{n}{k/6} \leq \binom{n}{k}.$$

Hence

$$\Pr\left(\exists S,\ |S| = k,\ |\Gamma(S) \setminus S| \leq \frac{k}{6}\right) \leq \left(\frac{k}{n}\right)^{dk/6} \binom{n}{k}^2.$$

For $d = 48$, we have $d/6 = 8$, so the last expression becomes

$$\left(\frac{k}{n}\right)^{8k} \binom{n}{k}^2.$$

We now show that this is at most $\binom{n}{k}^{-1}$. Indeed, it is enough to prove

$$\left(\frac{k}{n}\right)^{8k} \binom{n}{k}^3 \leq 1.$$

Using the standard estimate

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k,$$

we get

$$\left(\frac{k}{n}\right)^{8k} \binom{n}{k}^3 \leq \left(\frac{k}{n}\right)^{8k} \left(\frac{en}{k}\right)^{3k} = e^{3k} \left(\frac{k}{n}\right)^{5k}.$$

Since $k \leq n/2$, we have $k/n \leq 1/2$. Therefore,

$$e^{3k} \left(\frac{k}{n}\right)^{5k} \leq \left(\frac{e^3}{32}\right)^k.$$

Since

$$\frac{e^3}{32} < 1,$$

we conclude that

$$\left(\frac{k}{n}\right)^{8k}\binom{n}{k}^3 \le 1.$$

Hence

$$\left(\frac{k}{n}\right)^{8k}\binom{n}{k}^2 \le \binom{n}{k}^{-1}.$$

Therefore,

$$\Pr\left(\exists S,\ |S| = k,\ |\Gamma(S)\setminus S| \le \frac{k}{6}\right) \le \binom{n}{k}^{-1}.$$

Taking a union bound over all $2 \le k \le n/2$, we get

$$\Pr\left(\exists S,\ 0 < |S| \le \frac{n}{2},\ |\Gamma(S)\setminus S| \le \frac{|S|}{6}\right) \le \sum_{k=2}^{n/2}\binom{n}{k}^{-1}.$$

Singleton sets are not bad, since every vertex has degree d.

It remains to bound the sum. Since the binomial coefficients $\binom{n}{k}$ are increasing for $k \le n/2$, for every $3 \le k \le n/2$ we have

$$\binom{n}{k} \ge \binom{n}{3}.$$

Therefore,

$$\sum_{k=2}^{n/2}\binom{n}{k}^{-1} = \binom{n}{2}^{-1} + \sum_{k=3}^{n/2}\binom{n}{k}^{-1} \le \binom{n}{2}^{-1} + n\binom{n}{3}^{-1}.$$

Since

$$\binom{n}{2} = \Theta(n^2) \qquad \text{and} \qquad \binom{n}{3} = \Theta(n^3),$$

we get

$$\binom{n}{2}^{-1} + n\binom{n}{3}^{-1} = \mathcal{O}(n^{-2}).$$

Thus

$$\sum_{k=2}^{n/2}\binom{n}{k}^{-1} = \mathcal{O}(n^{-2}).$$

Therefore, with probability at least $1 - \mathcal{O}(n^{-2})$, every set $S \subseteq V$ with $0 < |S| \le n/2$ satisfies

$$|\Gamma(S)\setminus S| > \frac{|S|}{6}.$$

In particular,

$$|E(S, V\setminus S)| \ge |\Gamma(S)\setminus S| > \frac{|S|}{6}.$$

Consequently,

$$\phi(G) = \min_{0<|S|\le n/2}\frac{|E(S, V\setminus S)|}{d|S|} \ge \frac{1}{6d} = \frac{1}{288}.$$

□

References

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