Combinatorics of Computation Spring 2026

The Lovász Local Lemma

Prof. Jara Uitto *Antonio Cruciani*

Overview

A common way to prove the existence of a combinatorial object is to define a random experiment and show that the probability of obtaining an object with the desired properties is positive. This is the essence of the probabilistic method.

Often, we describe failure through a collection of *bad events*. If these bad events are independent and each occurs with small probability, then showing that none of them occurs is straightforward. More generally, even when the bad events are not independent, it may still be possible to prove that with positive probability none of them happens.

The Lovász Local Lemma is a fundamental tool for exactly this situation: it shows that a positive-probability outcome still exists when each bad event is unlikely and depends on only a limited number of other bad events. In this lecture we introduce the lemma, prove a standard symmetric form, and see several applications.

1 The Lovász Local Lemma

This is one of the most elegant and useful tools in applying the probabilistic method. Let us consider the following setting $E_1, \dots, E_n$ be a set of "bad events" in a probability space Ω. We want to show that the probability that no bad event occurs is positive.

This is easy if the events are mutually independent, i.e., for any $I \subseteq \{1, \dots, n\}$

$$\Pr\left(\bigcap_{i=1}^{n} E_i\right) = \prod_{i \in I} \Pr\left(E_i\right).$$

In that case, the events $\overline{E_i}$ are also independent and hence

$$\Pr\left(\bigcap_{i=1}^{n} \overline{E_i}\right) = \prod_{i \in I} \Pr\left(\overline{E_i}\right) = \prod_{i \in I} \left[1 - \Pr\left(E_i\right)\right] > 0.$$

assuming that $\Pr(E_i) < 1$ for all i.

However, in most cases, mutual independence can be too much to ask for. Lovász showed that less than mutual independence suffices to show that the probability of no bad event happening is positive. We will still have to assume some kind of independence, or better to say – limited dependence.

Definition 1. *An event A is mutually independent of the events $B_1, B_2, \ldots, B_n$ if for any subset $I \subseteq \{1, \ldots, n\}$, it holds that*

$$\Pr\left(A \mid \bigcap_{i \in I} B_i\right) = \Pr(A)$$

Now, we introduce a tool that will be used in what follows. Imagine to have a set of random variables, that might have some dependencies among each other. A useful way to visualize these dependencies could be to "draw" a graph in which nodes encodes the random variables and edges among nodes encode dependencies among the random variables. Formally, we define this object as a *dependency graph.*

Definition 2 (Dependency Graph)**.** *A dependency graph for a set of events $E_1, E_2, \ldots, E_n$ s a graph $G = (V, E)$ on vertex set $V = \{1, \ldots, n\}$ such that for all $1 \le i \le n$, E_i is mutually independent of the events $\{E_j \mid (i,j) \notin E\}$.*

Now we state the star of the lecture.

Theorem 3 (Lovász Local Lemma)**.** *Let $E_1, E_2, \ldots, E_n$ be events such that:*

- $\Pr(E_i) \le p$ *for all* i*,*
- *the degree of a dependency graph of these events is at most* d*,*
- $4dp \le 1$.

Then

$$\Pr\left(\bigcap_{i=1}^{n} \overline{E_i}\right) > 0.$$

Proof. If $d = 0$, then the events are mutually independent, so

$$\Pr\left(\bigcap_{i=1}^{n} \overline{E_i}\right) = \prod_{i=1}^{n} \Pr(\overline{E_i}) > 0.$$

So assume $d \ge 1$.

By Lemma 4, for every i we have

$$\Pr\left(E_i \,\middle|\, \bigcap_{j=1}^{i-1} \overline{E_j}\right) \le 2p.$$

Hence,

$$\begin{aligned}
\Pr\left(\bigcap_{i=1}^{n} \overline{E_i}\right) &= \prod_{i=1}^{n} \Pr\left(\overline{E_i} \,\middle|\, \bigcap_{j=1}^{i-1} \overline{E_j}\right) \\
&= \prod_{i=1}^{n} \left(1 - \Pr\left(E_i \,\middle|\, \bigcap_{j=1}^{i-1} \overline{E_j}\right)\right) \\
&\ge \prod_{i=1}^{n} (1 - 2p).
\end{aligned}$$

Since $d \geq 1$ and $4dp < 1$, we have $2p < 1$, so $(1-2p)^n > 0$. Therefore

$$\Pr\left(\bigcap_{i=1}^{n} \overline{E_i}\right) > 0. \qquad \square$$

Next, we need to prove the key lemma used in the proof above.

Lemma 4. *For all $S \subseteq \{1, \ldots, n\}$ and all $k \notin S$,*

$$\Pr\left(E_k \,\middle|\, \bigcap_{i \in S} \overline{E_i}\right) \leq 2p.$$

Proof. We prove the claim by induction on $|S|$.

If $|S| = 0$, then

$$\Pr(E_k \mid \bigcap_{i \in S} \overline{E_i}) = \Pr(E_k) \leq p \leq 2p.$$

Now assume $|S| > 0$. Let

$$S_1 = \{\, i \in S : i \text{ is adjacent to } k \text{ in the dependency graph} \,\}, \qquad S_2 = S \setminus S_1.$$

For any $T \subseteq \{1, \ldots, n\}$, define

$$F_T = \bigcap_{i \in T} \overline{E_i}.$$

If $S_1 = \emptyset$, then E_k is independent of all events in S, hence also of F_S, and therefore

$$\Pr(E_k \mid F_S) = \Pr(E_k) \leq p.$$

So suppose $S_1 \neq \emptyset$. Note that $|S_1| \leq d$ and $|S_2| < |S|$. Then

$$\Pr\left(E_k \mid F_S\right) = \frac{\Pr\left(E_k \cap F_S\right)}{\Pr(F_S)} = \frac{\Pr\left(E_k \cap F_{S_1} \mid F_{S_2}\right) \Pr\left(F_{S_2}\right)}{\Pr\left(F_{S_1} \mid F_{S_2}\right) \Pr\left(F_{S_2}\right)},$$

For the numerator,

$$\Pr(E_k \cap F_{S_1} \mid F_{S_2}) \leq \Pr(E_k \mid F_{S_2}) = \Pr(E_k) \leq p,$$

because E_k is independent of all events indexed by S_2.

For the denominator, by the union bound and the induction hypothesis,

$$\begin{aligned}
\Pr\left(F_{S_1} \mid F_{S_2}\right) &= \Pr\left(\bigcap_{i \in S_1} \overline{E_i} \mid F_{S_2}\right) \\
&\geq 1 - \sum_{i \in S_1} \Pr\left(E_i \mid F_{S_2}\right) \\
&\geq 1 - \sum_{i \in S_1} 2p \\
&\geq 1 - 2pd \\
&\geq \frac{1}{2}.
\end{aligned}$$

By using these bounds on the numerator and the denominator we get

$$\Pr\left(E_k \mid F_S\right) \leq \frac{p}{1/2} = 2p.$$

□

2 Using the Lovász Local Lemma

Here we show how to use this technique to some well known problems.

2.1 Edge-Disjoint Paths

Assume that n pairs of users need to communicate in a graph. Each pair $i \in \{1, \ldots, n\}$ can choose from a collection F_i of paths. For $i \neq j$, let $E_{\{i,j\}}$ be the event that the paths chosen by pairs i and j share an edge. Then $E_{\{i,j\}}$ is independent of all events $E_{\{i',j'\}}$ where $\{i,j\} \cap \{i',j'\} = \emptyset$. This means that each event ahs less than $2n$ neighbors in the dependency graph. Note that there are $\binom{n}{2} = \frac{n(n-1)}{2}$ events in total.

Now, assume that each F_i consists of m paths and that for any i and j: a path in F_i intersects with at most k paths in F_j. Then

$$\Pr\left(E_{\{i,j\}}\right) \leq \frac{k}{m}.$$

If d is the degree of the dependency graph, we have

$$d \leq 2n,$$

so

$$4dp < \frac{8nk}{m} \leq 1.$$

And, if $\frac{8nk}{m} \leq 1$, then there is a choice of paths such that the n paths are disjoint.

2.2 K-Satisfiability

Here we consider the following decision problem.

Input: a collection of clauses $C_1, C_2, \ldots, C_m$ over n boolean variables $(x_1, \ldots, x_n)$ in k-CNF, i.e., each C_i is a **OR** of k variables and $\phi = C_1 \wedge C_2 \wedge \ldots, \wedge C_m$.

Question: Find a truth assignment to $x_1, \ldots, x_n$ that satisfies ϕ, i.e., makes $\phi =$ **True**.

We can use the Lovász Local Lemma to show that, under certain settings, such an assignment exists.

Theorem 5. *Let $\phi = C_1 \wedge C_2 \wedge \ldots, \wedge C_m$, where each C_i has exactly k literals. If no variable appears in more than $T = 2^k/(4k)$ clauses then ϕ is satisfiable.*

Proof. Assign each variable independently and uniformly at random.

For each clause C_i, let

$$E_i = \{C_i \text{ is not satisfied}\}.$$

Since C_i has exactly k literals,

$$\Pr(E_i) = 2^{-k}.$$

Two events E_i and E_j are independent if the clauses C_i and C_j do not share any variable. If each variable appears in at most T clauses, then each clause shares a variable with at most

$$k(T-1)$$

other clauses. Hence the dependency degree satisfies

$$d \le k(T-1).$$

Therefore,

$$4pd \le 4 \cdot 2^{-k} \cdot k(T-1) < 4 \cdot 2^{-k} \cdot k \cdot \frac{2^k}{4k} = 1.$$

By the Lovász Local Lemma,

$$\Pr\left(\bigcap_{i=1}^{m} \overline{E_i}\right) > 0,$$

so there exists a satisfying assignment. □

2.3 Hypergraph Coloring

Here we consider the problem of assigning colors to an hypergraph. First of all, let us define an hypergraph.

Definition 6 (Hypergraph)**.** *A hypergraph is a pair $\mathcal{H} = (V, \mathcal{E})$, where V is the set of vertices and $\mathcal{E}$ is the set of* hyperedges*, where each hyper edge is a subset of V.*

Unlike ordinary graphs, a hyperedge can connect any number of vertices, not just two (see Figure 1).

We say that an hypergraph $\mathcal{H}$ is two colorable if there exist a 2-coloring of $V(\mathcal{H})$ such that no hyper edge is monochromatic. We now state another standard form of the Local Lemma.

Theorem 7 (Lovász Local Lemma – restated)**.** *Let $d \in \mathbb{N}$ and $p \in \mathbb{R}$ with*

$$ep(d+1) \le 1.$$

Let $E_1, E_2, \ldots, E_n$ be events. If

- $\Pr(E_i) \le p$ *for all* $1 \le i \le n$*, and*
- *The degree of their dependency graph is bounded by d*

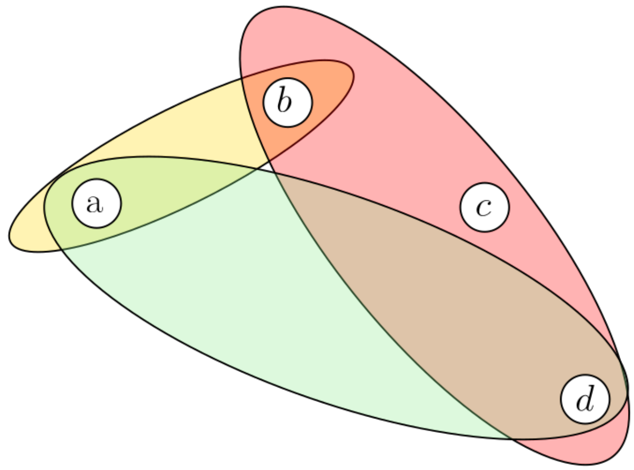

Figure 1: Hypergraph $\mathcal{H} = (V, \mathcal{E})$ where $V = \{a, b, c, d\}$ and $\mathcal{E} = \{\{a, b\}, \{b, c, d\}, \{a, b\}\}$.

then

$$\Pr\left(\bigcap_{i=1}^{n} \overline{E_i}\right) > 0.$$

Then we can state the following result

Theorem 8. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. Suppose that for every hyperedge $f \in \mathcal{E}$,*

- *$|f| \geq k$, and*
- *f intersects at most d other hyperedges.*

If

$$e(d + 1) \leq 2^{k-1},$$

then $\mathcal{H}$ is 2-colorable.

Proof. Color each vertex of $\mathcal{H}$ independently and uniformly at random with one of two colors.

For each hyperedge $f \in \mathcal{E}$, define the bad event

$$A_f = \{f \text{ is monochromatic}\}.$$

If none of the events A_f occurs, then no hyperedge is monochromatic, and hence the coloring is a proper 2-coloring of $\mathcal{H}$.

We now estimate the probability of a bad event. A hyperedge f is monochromatic if either all its vertices receive the first color or all its vertices receive the second color. Since the colors are chosen independently,

$$\Pr(A_f) = 2 \cdot \left(\frac{1}{2}\right)^{|f|} = \frac{2}{2^{|f|}} \leq \frac{2}{2^k} = 2^{1-k}.$$

Thus each bad event has probability at most

$$p = 2^{1-k}.$$

Next we describe the dependencies. The event A_f depends only on the colors assigned to the vertices of f. Therefore, if another hyperedge f' is disjoint from f, then A_f and $A_{f'}$ depend on disjoint sets of random choices, and so they are independent.

Hence A_f can only depend on events $A_{f'}$ such that $f' \cap f \neq \emptyset$. Define

$$D_f = \{ f' \in \mathcal{E} \setminus \{f\} : f' \cap f \neq \emptyset \}.$$

By assumption, $|D_f| \leq d$, so each bad event has at most d neighbors in the dependency graph.

We are therefore in the setting of the Lovász Local Lemma with

$$p = 2^{1-k} \qquad \text{and} \qquad d \leq d.$$

The condition

$$e(d+1) \leq 2^{k-1}$$

is equivalent to

$$e(d+1)2^{1-k} \leq 1,$$

that is,

$$ep(d+1) \leq 1.$$

Thus, by the Lovász Local Lemma, the probability that none of the bad events occurs is positive:

$$\Pr\left(\bigcap_{f \in \mathcal{E}} \overline{A_f} \right) > 0.$$

Therefore, there exists a 2-coloring of $\mathcal{H}$ in which no hyperedge is monochromatic. Hence $\mathcal{H}$ is 2-colorable. $\square$

References

[1] Mitzenmacher, Michael and Upfal, Eli, *Probability and computing: Randomization and probabilistic techniques in algorithms and data analysis, Cambridge university press, 2017*,