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# The Probabilistic Method

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## 1 Sample and Modify

So far we used the probabilistic method to directly construct the random structure with the desired property. Sometimes it is easier to first construct a random structure that does not have the required properties and then modify it such that it does.

### 1.1 Independent Sets

An independent set in a graph $G$ is a set of vertices with no edges between them. Finding the largest independent set in a graph is NP-hard. Again, the probabilistic method will allow us to give a bound on the size of the largest independent set.

**Theorem 1.** *A graph $G = (V, E)$ with $n$ vertices and $m \geq n/2$ edges has an independent set of size at least $\frac{n^2}{4m}$.*

*Proof.* Consider the following randomized algorithm: Denote by $d = \frac{2m}{n}$ the average degree of a node in $G$ and delete each vertex independently with probability $1 - \frac{1}{d}$. For each remaining edge, remove it and one of its adjacent vertices.

Note that after the first step, the resulting set of vertices does not necessarily form an independent set. However, after the second step, we do arrive at an independent set in $G$.

Let $X$ be the number of vertices surviving the first step. Then,

$$\mathbf{E}[X] = n \cdot \frac{1}{d} = \frac{n^2}{2m}.$$

Let $Y$ be the number of edges surviving the first step. Using linearity of expectation, we get:

$$\mathbf{E}[Y] = m \cdot \left(\frac{1}{d}\right)^2 = \frac{n^2}{4m}.$$

The second step removes the surviving edges and at most $Y$ vertices. Thus, the algorithm outputs an independent set of size at least $X - Y$, and so:

$$\mathbf{E}[X - Y] = \mathbf{E}[X] - \mathbf{E}[Y] = \frac{n^2}{2m} - \frac{n^2}{4m} = \frac{n^2}{4m}.$$

$\square$

### 1.2 Graphs with Large Girth

The *girth* of a graph is the length of its shortest cycle. If a graph is dense, we expect it to have a small girth. However, the probabilistic method allows us to show that there exist dense graphs with relatively large girth.

**Theorem 2.** *For every $k \geq 3$ and sufficiently large $n$, there exists a graph with $n$ vertices, $\frac{n^{1+\frac{1}{k}}}{4}$ edges, and girth at least $k$.*

*Proof.* Sample a random graph $G \in G(n, p)$ with $p = n^{-1+\frac{1}{k}}$, and let $X$ be the random variable that counts the number of edges in $G$. Then,

$$\mathbf{E}[X] = \binom{n}{2} \cdot p = \frac{n(n-1)}{2} n^{-1+\frac{1}{k}} = \left(\frac{n-1}{2}\right) n^{1+\frac{1}{k}-1} = \frac{1}{2}(n-1)n^{\frac{1}{k}} = \frac{1}{2}n^{1+\frac{1}{k}}\left(1 - \frac{1}{n}\right).$$

Let $Y$ be the number of cycles in $G$ of length at most $k-1$. There are at most $\binom{n}{\ell} \cdot \frac{(\ell-1)!}{2}$ candidate cycles of length $\ell$, and each one is present with probability $p^\ell$. So,

$$\mathbf{E}[Y] = \sum_{\ell=3}^{k-1} \binom{n}{\ell} \cdot \frac{(\ell-1)!}{2} \cdot p^\ell \leq \sum_{\ell=3}^{k-1} n^\ell \cdot p^\ell = \sum_{\ell=3}^{k-1} n^{\ell(1-1+\frac{1}{k})} = \sum_{\ell=3}^{k-1} n^{\ell/k} < kn^{(k-1)/k}.$$

Now modify the original graph $G$ by eliminating one edge from each cycle of length at most $k-1$. The modified graph therefore has girth at least $k$. The expected number of remaining edges is:

$$\mathbf{E}[X - Y] > \frac{n^{1+\frac{1}{k}}}{2} - kn^{(k-1)/k} \geq \frac{n^{1+\frac{1}{k}}}{4}$$

for sufficiently large $n$. $\square$

The key idea behind sample and modify proves existence by first generating a random object with good expected properties, then making small local changes to fix the remaining flaws.

## References


[1] Mitzenmacher, Michael and Upfal, Eli, *Probability and computing: Randomization and probabilistic techniques in algorithms and data analysis, Cambridge university press, 2017*,